*Journal Applied Mechanics and Technical Physics, Vol. 37, No. 1, 1996*

# ASYMPTOTIC BEHAVIOR OF A FLOW WITH A FREE SURFACE

**A. B. Usov**

UDC 532.526 : 532.516

In the present paper we construct the asymptotics of the solution of the plane problem of flow of a compressible, viscous fluid with a free surface. In [1–5] this problem is solved under additional assumptions of fluid incompressibility or stationarity, small flow viscosity, and smallness of tangent stresses. The case of an arbitrary tangent stress on the free surface of a compressible fluid and the nonstationary boundary layers arising in this case were not studied. The approach considered here allows one to construct the asymptotics of the solution at small time values without additional assumptions, with arbitrary tangent and normal stresses on the free surface. A boundary layer is constructed for stretching the spatial and time variables. We justify the choice of the stretching parameters and present a method for solving the boundary layer equations.

**1. Statement of the Problem.** Consider the plane isothermal problem of a compressible viscous fluid flow occupying the lower half-plane, with a free surface under the action of the given forces. The fluid flow in fixed Cartesian coordinates $xOz$ with center at an arbitrary point on the free surface of the fluid (Fig. 1) is described by the Navier–Stokes equations and boundary and initial conditions which, following [2; 6, p. 808; 7, p. 156], can be written in dimensionless form (at $\mu = \text{const}$) as

$$\mathrm{M}^2\Big(\frac{\partial \Pi}{\partial t} + v_x\frac{\partial \Pi}{\partial x} + v_z\frac{\partial \Pi}{\partial z}\Big) + \frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0,$$

$$\rho\Big(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_z\frac{\partial v_x}{\partial z}\Big) = -\rho\frac{\partial \Pi}{\partial x} + \rho F_x + \frac{1}{\mathrm{Re}}\Big(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial z^2} + \frac{1}{3}\frac{\partial}{\partial x}\Big(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z}\Big)\Big),$$

$$\rho\Big(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_z\frac{\partial v_z}{\partial z}\Big) = -\rho\frac{\partial \Pi}{\partial z} + \rho F_z + \frac{1}{\mathrm{Re}}\Big(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{3}\frac{\partial}{\partial z}\Big(\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z}\Big)\Big), \tag{1.1}$$

$$p = D\rho, \qquad \Pi = \int \frac{dp}{\rho(p)} = D\ln p;$$

$$v_x = v_z = 0, \quad \rho = \rho_*, \quad p = D\rho_*, \quad \Pi = D\ln(D\rho_*), \quad t = 0; \tag{1.2}$$

$$\mathbf{v} \to \mathbf{0}, \quad \nabla\cdot\mathbf{v} \to 0, \quad \rho \to \rho_*, \quad p \to D\rho_*, \quad x^2 + z^2 \to \infty; \tag{1.3}$$

$$p = T_n + \frac{2}{\mathrm{Re}}\Big(n_x^2\frac{\partial v_x}{\partial x} + n_z^2\frac{\partial v_z}{\partial z} + n_x n_z\Big(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\Big)\Big) - \frac{2}{3}\frac{1}{\mathrm{Re}}\operatorname{div}\mathbf{v} \quad \text{at} \quad z = \zeta(x,t),$$

$$\frac{1}{\mathrm{Re}}\Big((n_x\tau_z + n_z\tau_x)\Big(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\Big) + 2\Big(n_x\tau_x\frac{\partial v_x}{\partial x} + n_z\tau_z\frac{\partial v_z}{\partial z}\Big)\Big) = T_\varphi \quad \text{at} \quad z = \zeta(x,t); \tag{1.4}$$

$$\frac{d\zeta}{dt} = v_z, \quad \Delta = \sqrt{1 + (\partial\zeta/\partial x)^2}, \quad n_z = -1/\Delta, \quad n_x = \frac{\partial \zeta}{\partial x}\frac{1}{\Delta}, \quad \tau_z = -n_x, \quad ,\tau_x = n_z \ \text{at} \ z = \zeta(x,t). \tag{1.5}$$

Here $\mathbf{v} = (v_x, v_z)$ is the fluid particle velocity vector in the coordinates $xOz$; $p$ is the pressure; $\rho$ is the density; $\mathbf{F} = (F_x, F_z)$ is the vector of external mass forces acting on the fluid; Re is the Reynolds number; M is the Mach number; $D = \text{const} > 0$; $\Pi$ is a function of the pressure unified in the whole flow; $\rho_*$ is a given function;

Rostov State University, Rostov-on-Don 344006. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 37, No. 1, pp. 48–56, January–February, 1996. Original article submitted February 15, 1994; revision submitted January 9, 1995.

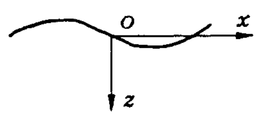


Fig. 1

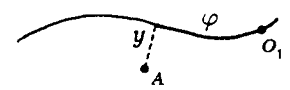


Fig. 2

$\mathbf{T} = (T_n, T_\varphi)$ is the vector of external forces acting on the free surface of the fluid $\Gamma$; $\mathbf{n} = (n_x, n_z)$, $\boldsymbol{\tau} = (\tau_x, \tau_z)$ are the normal and the tangent to $\Gamma$; $\zeta = \zeta(x,t)$ is the elevation of the free surface. Moreover

$$\mathrm{Re} = \rho^* v^* l^*/\mu, \quad c = \sqrt{D} = \mathrm{const}, \quad v^* = l^*/t^*, \quad \mathrm{M}^2 = (v^*)^2/c^2,$$

where $\rho^*$, $v^*$, $l^*$, and $t^*$ are the characteristic values of density, velocity, length, and time respectively; $\mu$ is the dynamical fluid viscosity; $c$ is the local sound propagation speed. Conditions (1.2) are the initial boundary conditions, (1.3) are the conditions of decay at infinity, (1.4) and (1.5) are the dynamic and kinematic conditions at the free surface.

The solution of the problem (1.1)–(1.5) depends on two parameters: M and Re. The asymptotics of the solution is constructed at large Reynolds numbers ($\mathrm{Re} \to \infty$). As for the Mach number, the following cases may arise:

the order of M depends on that of Re (at $\mathrm{Re} \to \infty$):

$$\mathrm{M}^2 = 0 \quad [(1/\sqrt{\mathrm{Re}})^k], \quad k \neq 0 \quad \text{at} \quad \mathrm{Re} \to \infty; \tag{1.6}$$

M does not depend on Re:

$$\mathrm{M}^2 = B(1/\sqrt{\mathrm{Re}})^0 \equiv B \equiv \mathrm{const}. \tag{1.7}$$

The choice between the cases influences the choice of stretching parameters of the boundary layer, which is discussed below. In the present work we consider the case of (1.7).

**2. Construction of the Functions of the First and the Second Iterative Process.** Following [8], we write the solution as

$$\mathbf{V} = \mathbf{V}^0 + \mathbf{V}^1, \quad \Pi = \Pi^0 + \Pi^1, \quad p = p^0 + p^1, \quad \rho = \rho^0 + \rho^1, \quad \zeta = \zeta^0 + \zeta^1, \tag{2.1}$$

where $\mathbf{V}^0$, $p^0$, $\Pi^0$, $\zeta^0$, and $\rho^0$ are the functions of the first iterative process, sought in the form of series

$$\begin{gathered}\mathbf{V}^0 = (v_x^0, v_z^0), \quad v_x^0 = \sum_{k=0}^{N} \varepsilon^k a_k(x,z,t), \quad \zeta^0 = \sum_{k=0}^{N} \varepsilon^k \zeta_k(x,t), \quad v_z^0 = \sum_{k=0}^{N} \varepsilon^k b_k(x,z,t), \\ p^0 = \sum_{k=0}^{N} \varepsilon^k p_k(x,z,t), \quad \rho^0 = \sum_{k=0}^{N} \varepsilon^k \rho_k(x,z,t), \quad \Pi^0 = \sum_{k=0}^{N} \varepsilon^k \Pi_k(x,z,t)\end{gathered} \tag{2.2}$$

($\varepsilon = 1/\sqrt{\mathrm{Re}}$ is a small parameter).

The equations for determining the functions of the first iterative processes are obtained by substituting series (2.2) into (1.1)–(1.5). In the leading-order approximation we get the Euler equations of an ideal, incompressible flow.

The functions $\mathbf{V}^1$, $p^1$, $\Pi^1$, $\zeta^1$, and $\rho^1$ of the second iterative process are defined in the vicinity of the free surface. To determine them, we introduce a local orthogonal system of coordinates $yO_1\varphi$ attached rigidly to the free surface $\Gamma$ (Fig. 2). As the origin (points $O_1$) an arbitrary point of $\Gamma$ is chosen; $y$ denotes the normal distance from a point, say $A$ (Fig. 2), to $\Gamma$, and $\varphi$ denotes the curve length along $\Gamma$. The functions $\mathbf{V}^1$, $p^1$, $\Pi^1$, $\zeta^1$, and $\rho^1$ are sought in the form of series

$$p^1 = \sum_{k=N_3}^{N} \varepsilon^k \beta_k^0(s,\varphi,\tau), \quad \rho^1 = \sum_{k=N_3}^{N} \varepsilon^k \alpha_k(s,\varphi,\tau), \quad \Pi^1 = \sum_{k=N_3}^{N} \varepsilon^k \beta_k(s,\varphi,\tau), \tag{2.3}$$

$$\zeta^1 = \sum_{k=N_4}^{N} \varepsilon^k \chi_k(x,\tau), \quad \mathbf{V}^1 = (v_y^1, v_\varphi^1), \quad v_y^1 = \sum_{k=N_1}^{N} \varepsilon^k h_k(s,\varphi,\tau), \quad v_\varphi^1 = \sum_{k=N_2}^{N} \varepsilon^k g_k(s,\varphi,\tau)$$

($N$, $N_1$, $N_2$, $N_3$, $N_4$ are integers).

The vectors $\mathbf{V}^0$ and $\mathbf{V}^1$ can also be represented as

$$\mathbf{V}^0 = (v_y^0, v_\varphi^0), \quad \mathbf{V}^1 = (v_x^1, v_z^1), \quad v_y^0 = \sum_{k=0}^{N} \varepsilon^k a_k^0(x,z,t),$$
$$v_\varphi^0 = \sum_{k=0}^{N} \varepsilon^k b_k^0(x,z,t), \quad v_x^1 = \sum_{k=N_2}^{N} \varepsilon^k h_k^0(s,\varphi,\tau), \quad v_z^1 = \sum_{k=N_2}^{N} \varepsilon^k g_k^0(s,\varphi,\tau). \tag{2.4}$$

The functions $\alpha_k$, $\beta_k$, $\beta_k^0$, $\chi_k$, $h_k$, and $g_k$ ($k = 0, 1, 2, \ldots, N$) of the boundary layer depend on the stretched variable

$$s = y/\varepsilon^{k_1} \tag{2.5}$$

and on the "fast" time

$$\tau = t/\varepsilon^{k_2} \tag{2.6}$$

($k_1$, $k_2$ are numbers, $k_1 > 0$, $k_2 \geqslant 0$).

The choice of the stretching parameters ($k_1$ and $k_2$) is based on the analysis of the flow equations written in the local coordinates. To construct the boundary layer it is required that the order of the term $(\partial^2 v_\varphi^1/\partial s^2)$ be not less than that of $(\partial \Pi^1/\partial\varphi)$, $(\mathbf{v}^1, \nabla)\mathbf{v}^1$, and $(\partial v_\varphi^1/\partial t)$ at $\varepsilon \to 0$. Otherwise, to determine the boundary layer correction terms one will obtain equations of no higher than the first order with respect to $s$. Since we have two boundary conditions for the function $v_\varphi^1$, a contradiction arises that makes the assumption unacceptable. Substituting the series (2.2)–(2.4) into the equations of motion requires satisfying the following inequalities (at $\varepsilon \to 0$):

$$\varepsilon^{N_2-k_2}\left|\frac{\partial g_{N_2}}{\partial\tau}\right| \leqslant \varepsilon^{2+N_2-2k_1}\left|\frac{\partial^2 g_{N_2}}{\partial s^2}\right|, \qquad \varepsilon^{N_2-k_1+N_1}\left|h_{N_1}\frac{\partial g_{N_2}}{\partial s}\right| \leqslant \varepsilon^{2+N_2-2k_1}\left|\frac{\partial^2 g_{N_2}}{\partial s^2}\right|,$$
$$\varepsilon^{2+N_2-2k_1}\left|\frac{\partial^2 g_{N_2}}{\partial s^2}\right| \geqslant \varepsilon^{N_3}\left|\frac{\partial\beta_{N_3}}{\partial\varphi}\right|, \qquad \varepsilon^{2+N_1-2k_1}\left|\frac{\partial^2 h_{N_1}}{\partial s^2}\right| \geqslant \varepsilon^{N_3-k_1}\left|\frac{\partial\beta_{N_3}}{\partial s}\right| \quad \text{if} \quad k_2 \neq 0,$$

that is, respectively,

$$2-2k_1 \leqslant -k_2, \quad 2-2k_1 \leqslant N_1-k_1, \quad 2-2k_1 \leqslant N_3, \quad N_1+2-2k_1 \leqslant N_3-k_1 \quad \text{if} \quad k_2 \neq 0. \tag{2.7}$$

To preserve the continuity equation in stretched variables it is necessary that the order of the terms $(\partial v_\varphi^1/\partial\varphi)$ and $(\partial v_y^1/\partial s)$ [$(\partial\Pi^1/\partial t)$ and $(\partial v_\varphi^1/\partial\varphi)$, $(\partial v_y^1/\partial s)$ at $k_2 \neq 0$] have an identical order of smallness (at $\varepsilon \to 0$). From here we find that

$$N_3 - k_2 = N_1 - k_1 \quad \text{if} \quad k_2 \neq 0; \qquad N_2 = N_1 - k_1. \tag{2.8}$$

We substitute the series (2.2)–(2.4) into the second boundary condition (1.4). In the resulting relation we expand the functions dependent on the "slow" time $t$ in a Taylor series at $t = 0$, and see that to satisfy the boundary condition in the leading-order approximation (at $\varepsilon \to 0$) the following equation should hold:

$$\varepsilon^{N_2-k_1}\frac{\partial g_{N_2}}{\partial s} + \varepsilon^{k_2}\tau\frac{\partial}{\partial t}\left(\frac{\partial a_0^0}{\partial y}\right)_{t=y=0} = 0 \quad \text{at} \quad T_\varphi \equiv 0,$$

or

$$\varepsilon^{N_2-k_1+2}\frac{\partial g_{N_2}}{\partial s} = \varepsilon^{k_2}\tau\left.\frac{\partial T_\varphi}{\partial t}\right|_{t=y=0} \quad \text{at} \quad T_\varphi \not\equiv 0,$$

i.e.,

$$N_2 = k_1 + k_2 \qquad \text{at} \quad T_\varphi \equiv 0; \tag{2.9}$$

$$N_2 = -2 + k_1 + k_2 \qquad \text{at} \quad T_\varphi \not\equiv 0. \tag{2.10}$$

The kinematic condition (1.5) implies that

$$N_4 - k_2 > \min(N_1, N_2).$$

Employing relations (2.7)–(2.10) we obtain the following possible versions of boundary layer stretching:

$$k_2 = 0, \quad k_1 = 4/3, \quad N_1 = 2/3, \quad N_3 = 0, \quad N_2 = -2/3, \tag{2.11}$$

and also a class of coinciding stretching parameters, which satisfy the relations

$$k_1 > 2, \quad k_2 = 2k_1 - 2, \quad N_2 = -4 + 3k_1, \quad N_1 = -4 + 4k_1, \quad N_3 = -6 + 5k_1; \tag{2.12}$$

$$k_2 = 2, \quad k_1 = 2, \quad N_1 = 4, \quad N_3 = 4, \quad N_2 = 2. \tag{2.13}$$

Solutions corresponding to (2.12) obey the same equations and yield the same result after switching to non-stretched coordinates. In what follows we construct the asymptotics (2.12) and (2.13) and prove their equivalence at small time values.

**3. The Case $k_1 > 2$, $k_2 = 2k_1 - 2$.** For example, consider a set of stretching parameters:

$$k_2 = 4, \quad k_1 = 3, \quad N_1 = 8, \quad N_3 = 9, \quad N_2 = 5.$$

We substitute (2.2)–(2.4) into the equations of motion, taking into account the results of the first iterative process. Expanding the functions dependent on $t = \tau\varepsilon^4$ and $y = s\varepsilon^3$ into the Taylor series $t = 0$ $(y = 0)$ and summing up terms of the same order with respect to $\varepsilon$ yields the boundary layer equations

at $\varepsilon^0$

$$\rho_* \frac{\partial g_5}{\partial \tau} = \frac{\partial^2 g_5}{\partial s^2}, \quad \rho_* \frac{\partial h_8}{\partial \tau} = \frac{4}{3}\frac{\partial^2 h_8}{\partial s^2},$$

$$\mathrm{M}^2\left(\frac{\partial \beta_9}{\partial \tau} + g_5 \frac{\partial \rho_0}{\partial \varphi}\bigg|_{y=t=0}\right) + \frac{\partial g_5}{\partial \varphi} + \frac{\partial h_8}{\partial s} = 0, \qquad \beta_9 = D \ln \beta_9^0, \qquad \beta_9^0 = D\alpha_9; \tag{3.1}$$

. . . . . . . . . . . . . . . . . . . .

at $\varepsilon^n$

$$\rho_* \frac{\partial g_{n+4}}{\partial \tau} = \frac{\partial^2 g_{n+4}}{\partial s^2} + E_1, \quad \rho_* \frac{\partial h_{n+7}}{\partial \tau} = \frac{4}{3}\frac{\partial^2 h_{n+7}}{\partial s^2} + E_2,$$

$$\mathrm{M}^2\left(\frac{\partial \beta_{n+8}}{\partial \tau} + g_{n+4} \frac{\partial \rho_0}{\partial \varphi}\bigg|_{y=t=0}\right) + \frac{\partial g_{n+4}}{\partial \varphi} + \frac{\partial h_{n+7}}{\partial s} = E_3, \qquad \beta_{n+8} = E_4, \qquad \beta_{n+8}^0 = D\alpha_{n+8}. \tag{3.2}$$

Here $H_1$ is the Lamé coefficient of the local system of coordinates $yO_1\varphi$: $H_1 = 1 + y\text{æ}(\varphi)$; $\text{æ}(\varphi)$ is the free surface curvature; the functions $E_1$, $E_2$, $E_3$, and $E_4$ are known from the solutions for $k = 0, 1, 2, \ldots, n-1$.

Substituting the series (2.2)–(2.4) into the boundary and the initial conditions we obtain at $\varepsilon^k$.

$$a_k = b_k = 0, \quad x^2 + z^2 \to \infty, \quad k = 0, 1, 2, \ldots; \tag{3.3}$$

$$\begin{cases} a_k = b_k = 0, & t = 0, \\ \rho_k = \begin{cases} 0, & k \neq 0, \\ \rho_*, & k = 0, \end{cases} \quad p_k = \begin{cases} 0, & k \neq 0, \\ D\rho_*, & k = 0, \end{cases} & t = 0; \end{cases} \tag{3.4}$$

$$g_k = h_{k+3} = \alpha_{k+4} = \beta_{k+4} = \beta_{k+4}^0 = 0, \quad k = 5, 6, \ldots, \quad t = 0; \tag{3.5}$$

$$p_k + \beta_k^0 = (T_n)_k + 2\left(\frac{\partial h_{k+1}}{\partial s} + \frac{\partial a_{k-2}^0}{\partial y}\right) - \frac{2}{3}\left(\frac{\partial g_{k-2}}{\partial \varphi} + \frac{\partial b_{k-2}^0}{\partial \varphi}\right)$$

$$+ \frac{\partial h_{k+1}}{\partial s} + \frac{\partial a_{k-2}^0}{\partial y} + \frac{1}{\text{æ}}(h_{k-2} + a_{k-2}^0), \quad y = s = 0; \tag{3.6}$$

$$\frac{\partial h_{k-2}}{\partial\varphi}+\frac{\partial g_{k+1}}{\partial s}+\sum_{4n+j=k-2}\frac{\partial}{\partial t}\left(\frac{\partial a_j^0}{\partial\varphi}+\frac{\partial b_j^0}{\partial y}+\frac{1}{æ}a_j^0\right)\tau^n\frac{1}{n!}=(T_\varphi)_k,\quad y=s=0, \tag{3.7}$$

where

$$(T_\varphi)_k=\begin{cases}0, & k\neq 4n,\\ \dfrac{\tau^{k/2}}{(k/2)!}\left.\dfrac{\partial^{k/2}T_\varphi}{\partial t^{k/2}}\right|_{t=0}, & k=4n,\end{cases}\qquad n=0,\,1,\,\ldots;$$

$$d\zeta_k/dt=b_k,\quad d\chi_k/d\tau=g_{k-4}^0,\quad y=s=0. \tag{3.8}$$

Condition (3.6) involves functions that depend both on the "slow"$t=\tau\varepsilon^4$ and "fast" time $\tau$. Let us split it into two conditions, one containing only the functions of the "slow" time $t$, and the other containing the rest of the functions. This results in

$$p_k=(T_n)_k+2\frac{\partial a_{k-2}^0}{\partial y}-\frac{2}{3}\left(\frac{\partial b_{k-2}^0}{\partial\varphi}+\frac{\partial a_{k-2}^0}{\partial y}+\frac{a_{k-2}^0}{æ}\right),\quad y=0; \tag{3.9}$$

$$\beta_k^0=2\frac{\partial h_{k+1}}{\partial s}-\frac{2}{3}\left(\frac{\partial g_{k-2}}{\partial\varphi}+\frac{\partial h_{k+1}}{\partial s}+\frac{1}{æ}h_{k-2}\right),\quad s=0. \tag{3.10}$$

The problem is solved by the following algorithm:

(1) The Euler equations are solved under the conditions (3.3), (3.4), (3.8), and (3.9) ($k=0$) for determining $\rho_0$, $\Pi_0$, $\zeta_0$, $a_0$, $b_0$, and $p_0$;

(2) The problem (3.1), (3.5), (3.7), (3.8), (3.10) ($k=2,\,4$) with the conditions of decay at infinity is used to determine $g_5$, $h_8$, $\alpha_9$, $\beta_9$, $\beta_9^0$, and $\chi_9$;

(3) The algorithm is repeated from step 1 with $k=k+1$.

Now we will turn our attention to the determination of the main terms of the boundary-layer corrections to the solution. Using Green's function, we write the solution of (3.1) as

$$g_5=-\frac{2}{\rho_*}\int_0^\tau t\left.\frac{\partial T_\varphi}{\partial t}\right|_{t=0}R(s,\tau-t)\,dt,\qquad h_8=-\frac{4}{3\rho_*}\int_0^\tau\left.\frac{\partial g_5}{\partial\varphi}\right|_{s=0}R(s,(\tau-t)4/3)\,dt, \tag{3.11}$$

where

$$R(s,r)=(4\pi r/\rho_*)^{(-1/2)}\exp(-s^2/(4r/\rho_*))\qquad(\pi=3.14159265\ldots). \tag{3.12}$$

**4. The Case $k_2=2$, $k_1=2$.** Proceeding as in Section 3 we obtain the problems

at $\varepsilon^0$

$$\begin{gathered}\rho_*\frac{\partial g_2}{\partial\tau}=\frac{\partial^2 g_2}{\partial s^2},\quad \rho_*\frac{\partial h_4}{\partial\tau}=-\rho_*\frac{\partial\beta_4}{\partial s}+\frac{4}{3}\frac{\partial^2 h_4}{\partial s^2},\\ \mathrm{M}^2\left(\frac{\partial\beta_4}{\partial\tau}+g_2\left.\frac{\partial\rho_0}{\partial\varphi}\right|_{y=t=0}\right)+\frac{\partial g_2}{\partial\varphi}+\frac{\partial h_4}{\partial s}=0,\qquad \beta_4=D\ln\beta_4^0,\qquad \beta_4^0=D\alpha_4;\end{gathered} \tag{4.1}$$

.....................

at $\varepsilon^n$

$$\begin{gathered}\rho_*\frac{\partial g_{n+2}}{\partial\tau}=\frac{\partial^2 g_{n+2}}{\partial s^2}+E_1,\quad \beta_{n+4}^0=D\alpha_{n+4},\\ \rho_*\frac{\partial h_{n+4}}{\partial\tau}=-\rho_*\frac{\partial\beta_{n+4}}{\partial s}+\frac{4}{3}\frac{\partial^2 h_{n+4}}{\partial s^2}+E_2,\quad \beta_{n+4}=E_4,\\ \mathrm{M}^2\left(\frac{\partial\beta_{n+4}}{\partial\tau}+g_{n+2}\left.\frac{\partial\rho_0}{\partial\varphi}\right|_{y=t=0}\right)+\frac{\partial g_{n+2}}{\partial\varphi}+\frac{\partial h_{n+4}}{\partial s}=E_3.\end{gathered} \tag{4.2}$$

Here $E_k$ ($k=1,\,2,\,3,\,4$) are known functions. The boundary conditions have the form (3.3)-(3.10).

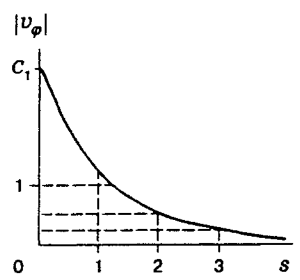


Fig. 3

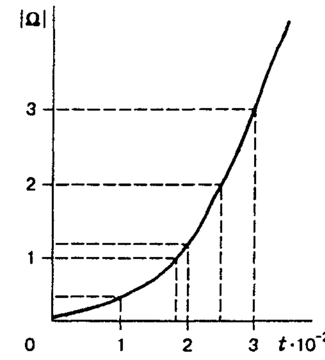


Fig. 4

Let us prove the equivalence of Eqs. (3.1), (3.2) and (4.1), (4.2) at small time values ($t < \varepsilon^4$). Employing the method of [9] we solve the problems (3.1) and (4.1). We make the change of variables

$$r = s/\sqrt{\tau}, \qquad \tau_1 = \sqrt{\tau}.$$

The solution is sought in the form

$$g_5(s,\varphi,\tau) = G_0(r,\varphi) + \tau_1 G_1(r,\varphi) + \dots, \quad h_8(s,\varphi,\tau) = H_0(r,\varphi) + \tau_1 H_1(r,\varphi) + \dots \, . \tag{4.3}$$

We substitute (4.3) into (3.1). Summing up coefficients of the same order of $\tau_1$, we have the problem for the first two nonzero terms of expansions (4.3)

$$G_2 - \frac{r}{2}\frac{\partial G_2}{\partial r} = \frac{1}{\rho_*}\frac{\partial^2 G_2}{\partial r^2}, \quad H_2 - \frac{r}{2}\frac{\partial H_2}{\partial r} = \frac{1}{\rho_*}\frac{\partial^2 H_2}{\partial r^2}. \tag{4.4}$$

Applying the same procedure to problem (4.1) yields exactly Eqs. (4.4). The boundary conditions of the two problems coincide. Consequently, after switching to the variables $y$, $\varphi$ the main terms of both asymptotics (2.12) and (2.13) at small time values [see (4.3)] coincide. Therefore, the two asymptotics are equivalent and we may confine ourselves only to those of Section 3.

**5. Tangent Stress on the Free Surface.** In order to illustrate the asymptotic theory developed in Section 3, consider an initially resting flow under the effect of tangent stress acting on its free boundary. Let

$$\begin{gathered} F_x = F_z = 0 \quad \text{in (1.1)}, \\ T_n = 0, \quad T_\varphi = f(t,\varphi) \quad \text{in (1.4)}, \\ f(0,\varphi) = 0, \quad \rho_* = \text{const} \quad \text{in (1.2)}. \end{gathered} \tag{5.1}$$

Here $f$ is a given function. In compliance with Sections 1–3 the leading terms of asymptotic expansion of the solution have the form

$$\begin{aligned} g_5(s/\varepsilon^3,\varphi,t/\varepsilon^4) &= -\frac{2}{\rho_*}\int\limits_0^{t/\varepsilon^4} \tau \left.\frac{\partial T_\varphi}{\partial t}\right|_{t=0} R(s, t/\varepsilon^4 - \tau)\,d\tau, \\ h_8(s/\varepsilon^3,\varphi,t/\varepsilon^4) &= -\frac{4}{3\rho_*}\int\limits_0^{t/\varepsilon^4} \left.\frac{\partial g_5}{\partial \varphi}\right|_{s=0} R(s,(t/\varepsilon^4 - \tau)4/3)\,d\tau, \\ \beta_9(s/\varepsilon^3,\varphi,t/\varepsilon^4) &= -\frac{1}{\mathrm{M}^2}\int\limits_0^{t/\varepsilon^4}\left( \left.\frac{\partial g_5}{\partial\varphi}\right|_{s=0} + \left.\frac{\partial h_8}{\partial s}\right|_{s=0} + \mathrm{M}^2 g_5 \left.\frac{\partial \rho_0}{\partial \varphi}\right|_{\substack{y=t=0\\ s=0}} \right) d\tau, \end{aligned} \tag{5.2}$$

$$\chi_9(\varphi, t/\varepsilon^4) = \int_0^{t/\varepsilon^4} g_5^0\Big|_{s=0} d\tau,$$

where $R$ is determined from (3.12).

Due to the splitting of the boundary condition (3.6) in (3.9) and (3.10), the functions of the first iterative process equal zero identically in any approximation. This is where it differs from an incompressible fluid [5]. The solution of the problem has a boundary-layer nature and reads as

$$\Pi = D\ln(D\rho_*) + \Pi^1 = D\ln(D\rho_*) + \varepsilon^9\beta_9 + \varepsilon^{10}\beta_{10} + \dots, \quad \mathbf{V} = (v_y, v_\varphi),$$

$$\zeta = \varepsilon^9\zeta_9 + \varepsilon^{10}\zeta_{10} + \dots, \quad v_y = v_y^1 = \varepsilon^8 h_8 + \varepsilon^9 h_9 + \dots, \quad v_\varphi = v_\varphi^1 = \varepsilon^5 g_5 + \varepsilon^6 g_6 + \dots .$$

Figure 3 shows a plot of the tangent component of the velocity vector (function $|v_\varphi^1|$) in a fixed point $\varphi = \varphi_*$, $t = t_*$. Here

$$C_1 = \frac{1}{\varepsilon\sqrt{\rho_*\pi}} \int_0^{t_*} \tau/\sqrt{t_* - \tau}\, d\tau.$$

The functions $v_y^1$ and $\Pi^1$ behave similarly to $v_\varphi^1$ (exponential decay when moving off the free boundary).

Figure 4 shows the curl of velocity (function $|\mathbf{\Omega}| = |\mathrm{rot}\,\mathbf{V}|$) in a fixed point $\varphi = \varphi_*$, $y = y_*$. The vorticity grows exponentially as the time increases. The plots in Figs. 3 and 4 are presented for $f(\varphi, t) = t$.

We are unable to apply the methods of [1–5] to develop an asymptotic theory for the problem (1.1)–(1.5) in the case (5.1). Further assumptions are necessary concerning fluid incompressibility, flow stationarity or small viscosity, and smallness of tangent stress [$f = O(1)$ at $\varepsilon \to 0$]. The methods considered in this work allowed us to present explicit formulas (5.2) for the main terms of asymptotic expansions of the solution at small $t$, without additional assumptions.

The author thanks Prof. E. N. Potetyunko for formulating the problem and discussing of the results.

## REFERENCES


1. V. A. Batishchev, "Asymptotics of nonuniformly heated free boundary of capillary fluid at high Marangoni numbers," *Prikl. Mat. Mekh.*, **53**, No. 3, 425–432 (1989).
2. V. A. Batishchev, "On the asymptotics of a low-viscous flow under the action of tangent stresses on the free boundary," *Zh. Prikl. Mekh. Tekh. Fiz.*, No. 5, 101–107 (1987).
3. V. A. Batishchev, "Boundary layers near the flat free boundary of fluid caused by axisymmetric tangent stresses," *Prikl. Mat. Mekh.*, **57**, No. 5, 60–67 (1993).
4. V. V. Pukhnachev "Boundary layers near free surfaces," in: *Computational and Asymptotics Methods for Boundary and Interior Layers*, Bool Press, Dublin (1982).
5. E. N. Potetyunko and L. S. Srubshchik, "Asymptotic analysis of wave flow with a free boundary," *Prikl. Mat. Mekh.*, **34**, No. 5, 45–52 (1970).
6. L. G. Loitsyanskii, *Fluid and Gas Mechanics* [in Russian], Nauka, Moscow (1970).
7. L. I. Sedov, *Continuum Mechanics* [in Russian], Nauka, Moscow (1970).
8. M. I. Vishik and L. A. Lyusternik, "Regular degeneration and boundary layer for linear differential equations with a small parameter," *Ukr. Mat. Zh.*, **12**, No. 5, 3–12 (1957).
9. M. Yu. Zhukov, A. A. Mkrtych'yan, Yu. M. Perevozkin, et al., "Thermoelastic dynamic problem for a ball coated with a multilayer shell," *Izv. Sev.-Kavk. Nauchn. Tsentera Vyssh. Shk. Estestv. Nauk*, No. 3, 27–32 (1976).
10. N. E. Kochin, I. A. Kibel', and N. V. Rose, *Theoretical Hydromechanics* [in Russian], Moscow (1963), Parts 1, 2.